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SBU AMS Quantitative Finance Qualifying Exam

May 14, 2021

Problem 1. (20p) (Option Pricing in a Binomial Model)

Consider a binomial tree model for an underlying asset process

$$\{S_n : 0 \le n \le N\}$$

where $S_0 = 4$. Let

$$S_{n+1} = \begin{cases} uS_n & \text{with probability } \widetilde{p} \\ dS_n & \text{with probability } 1 - \widetilde{p} \end{cases}$$

where $u = 2$ and $d = 1/u$. Assume that the one period risk-free interest rate is $r = 25\%$, and using $N = 2$ (i.e., $n = 0, 1, 2$) period binomial tree model find

(i) (5p) the $n = 0$ price V_0^C of an American call option that has intrinsic value $g_C(s) = (s - K)^+$, and the strike $K = 5$.

(ii) (5p) the $n = 0$ price V_0^P of an American put option that has intrinsic value $g_P(s) = (K - s)^+$, and the strike $K = 5$.

(iii) (5p) the $n = 0$ price V_0^S of an American straddle option that has intrinsic value $g_S(s) = g_C(s) + g_P(s)$, and both of the strikes are $K = 5$.

(iv) (5p) Explain why $V_0^S \le V_0^C + V_0^P$.

Problem 2. (20p) (Brownian Motion and Time Inversion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let B_t, for $t \geq 0$, be a Brownian motion.

(i) (5p+5p=10p)
Show that

$$Cov(B_s, B_t) = \min(t, s),$$

and that

$$Corr(B_s, B_t) = \sqrt{\frac{\min(t, s)}{\max(t, s)}}.$$

(Hint: Use the properties in the definition of the Brownian motion.)

(ii) (10p)
Show that the following process defined via time inversion

$$W_t = \begin{cases} 0 & \text{if } t = 0 \\ tB_{\frac{1}{t}} & \text{if } t \neq 0 \end{cases}$$

is also a Brownian motion.
(Hint: Show the three properties from one of the definitions of the Brownian motion, use the first result from (i) above.)

Problem 3 (20p) (Digital Option - Probabilistic Approach)

Let $\{W_t : t \geq 0\}$ be a $\mathbb{P}$-standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the stock price S_t follow a GBM with the following SDE

$$dS_t = S_t \mu dt + S_t \sigma dW_t,$$

where μ is the drift parameter, $\sigma > 0$ is the volatility parameter, and let $r > 0$ denote the risk-free interest rate.

A digital (or cash-or-nothing) call option is a contract that pays \$1 at expiry time T if the spot price $S_T > K$ and nothing if $S_T \leq K$. In contrast, a digital (or cash-or-nothing) put pays \$1 at expiry time T if the spot price $S_T < K$ and nothing if $S_T \geq K$.

(i) (5p) Find an equivalent martingale measure $\mathbb{Q}$ under which the discounted stock price $e^{-rt}S_t$ is a martingale (discuss why this is a martingale).

(ii) (5p) By denoting $C_d(S_t, t; K, T)$ and $P_d(S_t, t; K, T)$ as the digital call and put option prices, respectively at time t, for $t < T$ show that

$$C_d(S_t, t; K, T) = e^{-r(T-t)} \mathbb{Q}\left(S_T > K \mid \mathcal{F}_t\right)$$

and

$$P_d(S_t, t; K, T) = e^{-r(T-t)} \mathbb{Q}\left(S_T < K \mid \mathcal{F}_t\right).$$

(iii) (5p) Using Ito lemma find the distribution of S_T given $\mathcal{F}_t$ under $\mathbb{Q}$, and show, using the risk-neutral valuation approach from (i) and (ii) above, that

$$C_d(S_t, t; K, T) = e^{-r(T-t)}\Phi(d_-) \quad \text{and} \quad P_d(S_t, t; K, T) = e^{-r(T-t)}\Phi(-d_-),$$

where

$$d_- = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du.$$

(iv) (5p) Verify that the put-call parity for a digital option is

$$C_d(S_t, t; K, T) + P_d(S_t, t; K, T) = e^{-r(T-t)}.$$

Problem 4. (20p) (Identification of ARMA Processes)

Figure 1 below shows simulated paths for various time series processes as well as the corresponding sample autocorrelation function (ACF) and partial autocorrelation function (PACF) plots. In each case the innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ are from a strict white noise process and the time series $(X_t)_{t\in\mathbb{Z}}$ follows one of the following models:

(a) (4p) $X_t = \varepsilon_t - 0.3\varepsilon_{t-1} - 0.4\varepsilon_{t-2},\ t \in \mathbb{Z}$

(b) (4p) $X_t = 10 + 20t + 2t\sin(t) + \varepsilon_t,\ t \in \mathbb{Z}$

(c) (4p) $X_t - 0.3X_{t-1} - 0.4X_{t-2} = \varepsilon_t,\ t \in \mathbb{Z}$

(d) (4p) $X_t = \varepsilon_t - 0.7\varepsilon_{t-1},\ t \in \mathbb{Z}$

(e) (4p) $X_t - 0.3X_{t-1} = \varepsilon_t - 0.6\varepsilon_{t-1},\ t \in \mathbb{Z}$

Match each model with the pictures, explaining your reasoning.

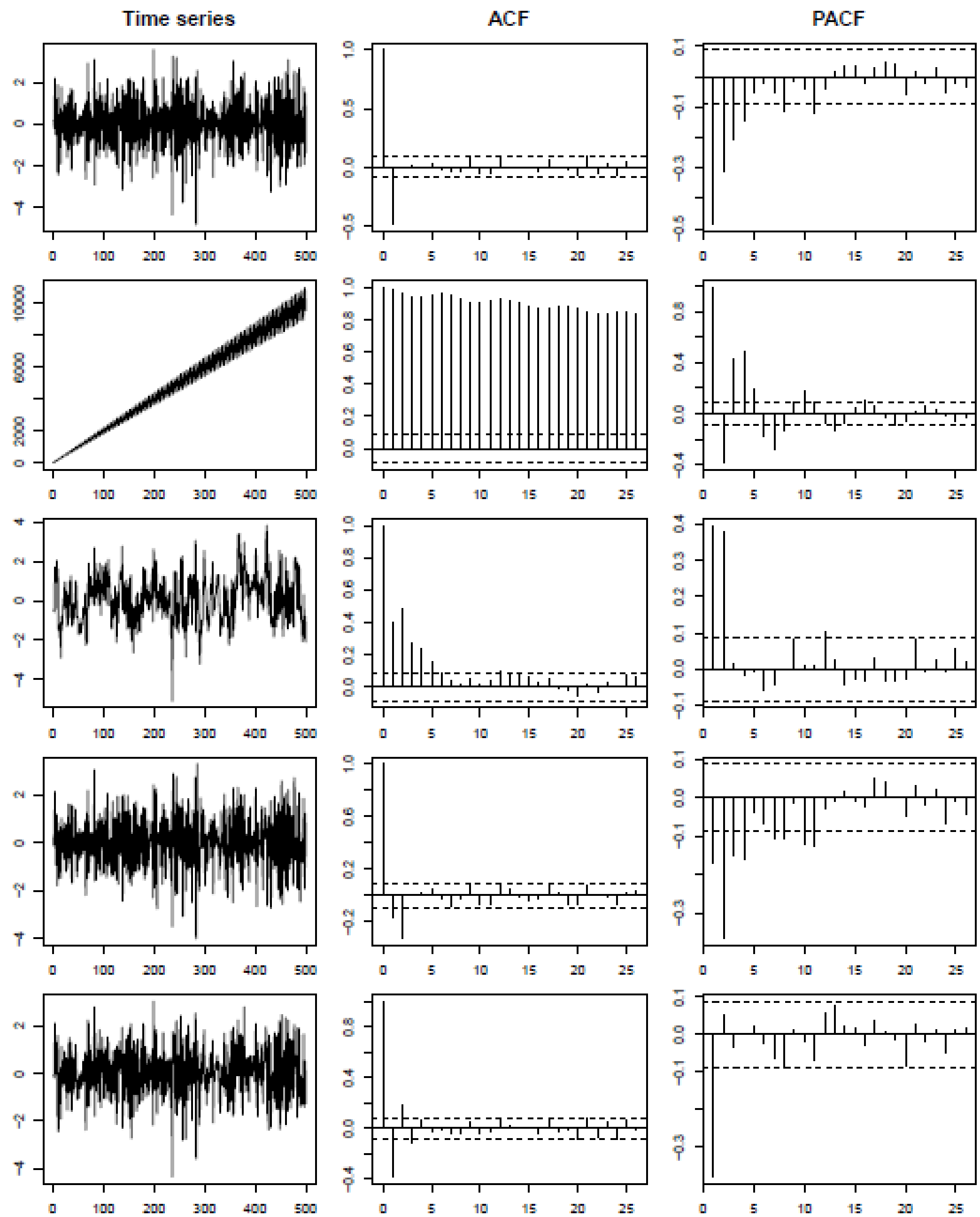

Figure 1: Paths of five time series $(X_t)_{t\in\mathbb{Z}}$ (left) with corresponding ACF (middle) and PACF (right).

Problem 5. (20p) ((Non)-Subadditivity of Value-at-Risk (VaR))

(i) (10p) Consider a portfolio of $d = 100$ defaultable corporate bonds. We assume that defaults of different bonds are independent; the default probability is identical for all bonds and is equal to 2%. The current price of the bonds is 100 USD. If there is no default, a bond pays in $t + 1$ (one year from now, say) an amount of 105 USD; otherwise there is no repayment. Hence L_i, the loss of bond i, is equal to 100 USD when the bond defaults and to -5 USD otherwise. Denote by Y_i the default indicator of firm i, i.e. Y_i is equal to one if bond i defaults in $[t, t+1]$ and equal to zero otherwise. We get $L_i = 100Y_i - 5(1 - Y_i) = 105Y_i - 5$ USD. Hence the L_i form a sequence of iid rvs with $\mathbb{P}(L_i = 5) = 0.98$ and $\mathbb{P}(L_i = 100) = 0.02$. Your goal is to compare two portfolios in terms of their VaR values, both with current value equal to 10000 USD.

- Portfolio A is fully concentrated and consists of 100 units of bond one.
- Portfolio B is completely diversified: it consists of one unit of each of the bonds.

Economic intuition suggests that portfolio B is less risky than portfolio A and hence should have a lower VaR. Compute VaR at a confidence level of 95% for both portfolios. Is the VaR subadditive in this example? (For a binomial distribution $M \sim B(100, 0.02)$, $\mathbb{P}(M \leq 5) \approx 0.984$ and $\mathbb{P}(M \leq 4) \approx 0.949$.)

(ii) (10p) Suppose $\mathbf{X} = (X_1, \ldots, X_d) \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\mathcal{M}$ denote the space of all risks of the form $L = \boldsymbol{\lambda}'\mathbf{X} = \sum_{j=1}^{d} \lambda_j X_j$. Prove that VaR_α is subadditive on $\mathcal{M}$ for all $\alpha \in [1/2, 1)$.
Hints: Without loss of generality you can assume $d = 2$, use the fact that VaR is translation invariant and positively homogeneous, i.e., $VaR_\alpha(\mu + \sigma X) = \mu + \sigma VaR_\alpha(X)$, and apply Cauchy-Schwarz inequality to get the result.

Problem 6. (20p) (An Interpretation of the Principal Components Transform)

Let $\mathbf{X}$ be a random vector with $\mathbb{E}(\mathbf{X}) = 0$ and $cov(\mathbf{X}) = \mathbf{\Sigma}$. Let $\mathbf{Y}$ be the vector given by the principal components transform of $\mathbf{X}$. In other words, $\mathbf{Y} = \mathbf{\Gamma X}$, where $\mathbf{\Sigma} = \mathbf{\Gamma \Lambda \Gamma}'$, the columns of $\mathbf{\Gamma}$ contain the orthonormal eigenvectors of $\mathbf{\Sigma}$ and $\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_d)$, where $\lambda_1 \geq \cdots \geq \lambda_d$ denote the sorted eigenvalues of $\mathbf{\Sigma}$. The jth principal component Y_j is thus $Y_j = \boldsymbol{\gamma}_j' \mathbf{X}$, where j is the jth column of $\mathbf{\Gamma}$ (the eigenvector corresponding to the jth largest eigenvalue of $\mathbf{\Sigma}$), also known as jth vector of loadings or jth principal axis.

(i) (10p) Show that the first principal component Y_1 of $\mathbf{X}$ satisfies

$$Var(Y_1) = \max_{\mathbf{a} \in \mathbb{R}^d, \|\mathbf{a}\|=1} Var(\mathbf{a}'\mathbf{X}),$$

that is Y_1 is the standardized linear combination $\mathbf{a}'\mathbf{X}$ of $\mathbf{X}$ with maximal variance among all linear combinations.

(ii) (10p) Show that the second principal component Y_2 of $\mathbf{X}$ satisfies

$$Var(Y_2) = \max_{\mathbf{a} \in \mathbb{R}^d,\ \|\mathbf{a}\|=1,\ \mathbf{a}'\boldsymbol{\gamma}_1=0} Var(\mathbf{a}'\mathbf{X}),$$

that is Y_2 is the standardized linear combination $\mathbf{a}'\mathbf{X}$ of $\mathbf{X}$ with maximal variance among all linear combinations orthogonal to the first principal axis.

Hint. Apply the method of Lagrange multipliers. It may also be useful to use the facts that $\frac{\partial}{\partial \mathbf{x}} \mathbf{a}'\mathbf{x} = \frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{a} = \mathbf{a}$, and $\frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{x} = (\mathbf{A} + \mathbf{A}')\mathbf{x}$.